Summary of nonlinear continuum mechanics

1. Fundamentals of continuum mechanics

The subject of continuum mechanics includes (comprises) the following basic ingredients

- the study of motion and deformation
- the study of stress in a continuum, and
- the mathematical description of the fundamental laws of physics governing the motion of a continuum.

1.1 Scalars: A physical quantity, completely described by a single real number, such as mass, density or temperature, is called scalar.

1.2 Vectors: Certain physical quantities, such as force, velocity or displacement, which posses both magnitude and direction, may be represented in a three-dimensional space by directed line segments that obey the parallelogram law of addition. We have some kind of vectors, such as free vector, bounded vector (can freely move), reference vector and loading vector (fixed point).

1.3 Tensors: A tensor is mathematical entity with certain well-defined transformation properties under a change of coordinates. The concept of tensor is a generalization of the familiar quantities scalar and vector.

Tensors of the order one are called vectors. Tensors of the order zero are called scalars.

Tensor of order n may be expressed in the form $A_{i_1,i_2\ldots i_n}\, e_{i_1} \otimes e_{i_2} \otimes \ldots \otimes e_{i_n}$

1.4 Operations (calculuses) of vectors and tensors

- Dot product (scalar product) of vectors **a** and **b**, denoted by **a**·**b** , produces a new scalar.

  Dot product of two order tensors **A** and **B**, denoted by **AB** $\{(\mathbf{AB})_{ij} = A_{ik}B_{kj}\}$, is again a second order tensor.

- Cross product (vector product) of vectors **a** and **b**, denoted **a** x **b**, produces a new vector.
- Tensor product (dyad product) of the vectors **a** and **b** is denoted by **a** ⊗ **b**, produces a second-order tensor.

Notation of tensor of the first order		Notation of tensor of the second order	
Indicial notation	$c_{ij} = a_i b_j$,	Indicial notation	$c_{ijkl} = a_{ij} b_{kl}$,
Symbolic notation	$\mathbf{C} = \mathbf{a} \otimes \mathbf{b}$,	Symbolic notation	$\mathbf{C} = \mathbf{A} \otimes \mathbf{B}$
Matrix algebra notation	$[C] = \{a\}\{b\}^T$	Matrix algebra notation	

- The **Trace** of a tensor **A** is a scalar denoted by $\mathrm{tr}\mathbf{A} = A_{ii}$,
- The **contraction** of tensors **A** and **B** is scalar denoted by $\mathbf{A} : \mathbf{B} = \mathrm{tr}(\mathbf{A}^T\mathbf{B}) = \mathrm{tr}(\mathbf{B}^T\mathbf{A}) = \mathbf{B} : \mathbf{A}$

1.5 Eigenvalues and eigenvectors of tensors

The scalars λ_i characterize eigenvalues (or principle values) of a tensor **A** if there exits corresponding nonzero normalized eigenvectors $\hat{n}_i$ (or principle directions) of **A**, so that

$$\mathbf{A}\,\hat{n}_i = \lambda_i\,\hat{n}_i \quad (i = 1, 2, 3)$$

A set of homogeneous equations for the unknown eigenvalues and eigenvectors is

$$(\mathbf{A} - \lambda_i\,\mathbf{I})\,\hat{n}_i = 0$$

For the above system to have solutions $\hat{n}_i \neq 0$ the determinant of the system must vanish. Thus

$$\mathrm{Det}(\mathbf{A} - \lambda_i \mathbf{I}) = 0$$

This requirement that we solve a cubic equation in λ (**characteristic equation**), is usually written as

$$\lambda^3 - I_1\lambda^2 + I_2\lambda - I_3 = 0$$

Here, $I_i(\mathbf{A})$, i = 1, 2, 3, are the so-called principle scalar invariants of **A**. In term of **A** and its principle values λ_i, i = 1, 2, 3

$$I_1(\mathbf{A}) = A_{ii} = \mathrm{tr}\mathbf{A} = \lambda_1 + \lambda_2 + \lambda_3$$

$$I_2(\mathbf{A}) = \frac{1}{2}(A_{ii}A_{jj} - A_{ji}A_{ij}) = \frac{1}{2}[(\mathrm{tr}\mathbf{A})^2 - \mathrm{tr}(\mathbf{A}^2)] = \mathrm{tr}\mathbf{A}^{-1}\det A = \lambda_1\lambda_2 + \lambda_1\lambda_3 + \lambda_2\lambda_3$$

$$I_3(\mathbf{A}) = \varepsilon_{ijk}A_{1i}A_{2j}A_{3k} = \det\mathbf{A} = \lambda_1\lambda_2\lambda_3$$

1.6 Transformation laws for the basis vectors and components

- Vectorial transformation law

We consider any vector u resolved along the two sets $\{\tilde{e}_i\}$ and $\{e_i\}$ of basis vectors, i.e.

$$\tilde{u}_i = \mathbf{u} \cdot \tilde{e}_i \quad ; \quad u_i = \mathbf{u} \cdot e_i$$

We assume that the relation between the basis vectors $\tilde{e}_i$ and e_i is known by the cosines Q_{ij}, we have the vectorial transformation law for the Cartesian components of vector u, i.e.

$$[\tilde{u}] = [\mathbf{Q}]^T [\mathbf{u}] \quad , \quad [\mathbf{u}] = [\mathbf{Q}] [\tilde{u}]$$

- Tensorial transformation law

$$[\tilde{A}] = [\mathbf{Q}]^T[\mathbf{A}][\mathbf{Q}] \quad \text{or} \quad \tilde{A}_{ij} = Q_{ki}Q_{mj}A_{km}$$

$$[\mathbf{A}] = [\mathbf{Q}][\tilde{A}][\mathbf{Q}]^T \quad \text{or} \quad A_{ij} = Q_{ik}Q_{jm}\tilde{A}_{km}$$

1.7 Gradient and related operators

Gradient of the scalar field $\Phi(x)$ may be written in the form

$$\mathrm{grad}\Phi = \nabla\Phi = \frac{\partial\Phi}{\partial x_i} e_i = \frac{\partial\Phi}{\partial x_1} e_1 + \frac{\partial\Phi}{\partial x_2} e_2 + \frac{\partial\Phi}{\partial x_3} e_3$$

in which $\nabla(\bullet)$ is a vector operator (or Nabla operator) of vector calculus. We further introduce the dot product, cross product and tensor product of vector operator ∇

$$\nabla \cdot (\bullet) = \frac{\partial(\bullet)}{\partial x_i} \cdot e_i \quad , \quad \nabla \times (\bullet) = e_i \times \frac{\partial(\bullet)}{\partial x_i} \quad , \quad \nabla \otimes (\bullet) = \frac{\partial(\bullet)}{\partial x_i} \otimes e_i$$

Divergence of a vector field **u**(x)

$$\mathrm{Div}\mathbf{u} = \nabla \cdot \mathbf{u} = \frac{\partial u_j}{\partial x_i} \mathbf{e}_j \cdot \mathbf{e}_i = \frac{\partial u_j}{\partial x_i} \delta_{ji} = \frac{\partial u_i}{\partial x_i} = \frac{\partial u_1}{\partial x_1} + \frac{\partial u_2}{\partial x_2} + \frac{\partial u_3}{\partial x_3}$$

Curl of a vector field

$$\mathrm{Curl}\mathbf{u} = \nabla \times \mathbf{u} = \frac{\partial u_j}{\partial x_i} \mathbf{e}_i \times \mathbf{e}_j = \varepsilon_{ijk} \frac{\partial u_j}{\partial x_i} e_k$$

Gradient of a vector field

$$\mathrm{Grad}\mathbf{u} = \nabla \otimes \mathbf{u} = \frac{\partial u_i}{\partial x_j} \mathbf{e}_i \otimes \mathbf{e}_j \quad \text{with Cartesian components } (\mathrm{grad}\mathbf{u})_{ij} = \frac{\partial u_i}{\partial x_j}$$

Divergence and gradient of a (second-order) tensor field

$$\mathrm{Div}\mathbf{A} = \nabla \cdot \mathbf{A} = \frac{\partial A_{ij}}{\partial x_j}\mathbf{e}_i$$

$$\mathrm{Grad}\mathbf{A} = \nabla \otimes \mathbf{A} = \frac{\partial A_{ij}}{\partial x_k}\mathbf{e}_i \otimes \mathbf{e}_j \otimes \mathbf{e}_k$$

Laplacian operator

$$\nabla^2(\bullet) = \nabla \cdot \nabla(\bullet) = \nabla \cdot \frac{\partial(\bullet)}{\partial x_i}\mathbf{e}_i = \frac{\partial^2(\bullet)}{\partial x_i^2}$$

1.8 Integral theorems

Divergence theorem. Suppose **u**(**x**) and **A**(**x**) are any smooth vector and tensor fields defined on some convex three-dimensional region in physical space with volume v, and on a closed surface s bounding this volume.

$$\int_s \mathbf{u}\cdot\mathbf{n}\,ds = \int_v \mathrm{div}\,\mathbf{u}\,dv \quad , \quad \int_s A\mathbf{n}\,ds = \int_v \mathrm{div}\,\mathbf{A}\,dv$$

Stokes' theorem. It relates a surface integral, which is valid over any open surface s, to a line integral around the bounding closed curve c in three-dimensional space.

$$\oint_c \mathbf{u}\cdot d\mathbf{x} = \int_s \mathrm{curl}\,\mathbf{u}\cdot\mathbf{n}ds$$

2. Deformation gradient

Consider a continuous body with **reference (undeformed) configuration** Ω_0 and reference time t=0. Then, an assumed motion χ maps the reference configuration to the current configuration Ω at time t. Hence, a point $X \in \Omega_0$ transforms to a place $x \in \Omega$, where X and x (with material and spatial coordinates X_1, X_2, X_3 and x_1, x_2, x_3) characterize positions of a particle in the reference and current configuration relative to fixed set of axes.

The deformation is represented by the mapping $\chi : \Omega_0 \rightarrow \Omega$

$$x = \chi(X, t) \tag{1}$$

The mapping χ is called the deformation from Ω_0 to Ω, and χ is required to be one-to-one and to satisfy appropriate regularity conditions.

A motion χ of a body will generally change its shape, position and orientation. A continuum body which is able to change its shape is said to be **deformable**.

The so-called **material description** is a characterization of the motion with respect to the material coordinates (X_i) and time t. In the material description attention is paid to a particle, we observe what happens to the particle as it moves.

The so-called **spatial description** is a characterization of the motion with respect to the spatial coordinates (x_i) and time t. In the spatial description attention is paid to point of space, we study what happens to the point as time changes.

The quantity **F** is crucial in **nonlinear continuum mechanics** and is a primary measure of deformation

$$\mathbf{F} = \frac{\partial x}{\partial X} = \frac{\partial \chi(X,t)}{\partial X} \tag{2}$$

called the **deformation gradient**.

The determinant of deformation gradient, known as **Jacobian determinant** (volume ratio) will be denoted

$$J = \det \mathbf{F}(\mathbf{X}, t) = \frac{dv}{dV} \tag{3}$$

(It is the change in volume between reference and the current configuration at time t)

Displacement gradient tensor in the material description

$$\text{Grad}\mathbf{U} = \text{Grad}\mathbf{x}(\mathbf{X}, t) - \text{Grad}\mathbf{X} = \mathbf{F}(\mathbf{X}, t) - \mathbf{I}$$

3. Strain tensors

We describe a material line element with vector a_0 at the referential position X (which may be imagined as a **fiber**). The corresponding in the current configuration, a_0 is mapped onto a as

$$\mathbf{F}a_0 = a \tag{4}$$

The strain that results from the finite motions is obtained by measuring the change in the magnitude of any vector (λ is a stretch of the fiber)

$$\begin{aligned} \lambda a.\lambda a - a_0.\, a_0 &= (\mathbf{F}a_0)\cdot(\mathbf{F}a_0) - a_0\cdot a_0 \\ &= a_0\cdot(\mathbf{F}^T\mathbf{F})\, a_0 - a_0\cdot \mathbf{I}\, a_0 \\ &= a_0\cdot(\mathbf{C} - \mathbf{I})\, a_0 = 2\, a_0\cdot\mathbf{E}\, a_0 \end{aligned} \tag{5}$$

Implicit in this derivation is definition of the **right Cauchy-Green strain tensor C** and **Green-Lagrange strain tensor E** as

$$\mathbf{C} = \mathbf{F}^T\mathbf{F} \quad , \quad \mathbf{E} = \frac{1}{2}(\mathbf{C} - \mathbf{I}) \tag{6}$$

An addition strain measure is the **left Cauchy-Green strain tensor b** given as

$$\mathbf{b} = \mathbf{F}\,\mathbf{F}^T$$

We can decomposite the deformation gradient **F** and some strain tensors into multiplication of spherical (dilatational) and distortional parts

$$\mathbf{F} = (J^{1/3}\mathbf{I})\,\bar{\mathbf{F}} \tag{8}$$

$$\mathbf{C} = \mathbf{F}^T\mathbf{F} = J^{2/3}\,\bar{\mathbf{C}}, \qquad \bar{\mathbf{C}} = \bar{\mathbf{F}}^T\bar{\mathbf{F}} \tag{9}$$

$$\mathbf{b} = \mathbf{F}\,\mathbf{F}^T = J^{2/3}\,\bar{\mathbf{b}}, \qquad \bar{\mathbf{b}} = \bar{\mathbf{F}}\,\bar{\mathbf{F}}^T \tag{10}$$

$$\mathbf{E} = \frac{1}{2}(\mathbf{C} - \mathbf{I}) = J^{2/3}\,\bar{\mathbf{E}} + \frac{1}{2}(J^{1/3} - 1)\,\mathbf{I}, \qquad \bar{\mathbf{E}} = \frac{1}{2}(\bar{\mathbf{C}} - \mathbf{I}) \tag{11}$$

Push-forward and pull-back operations

The transformations between material and spatial quantities are typically called the **push-forward** operation and **pull-back** operation. In particular, a push-forward is an operation which transform a vector or tensor-valued quantity based on the reference configuration to the current configuration. A pull-back is an inverse operation, which transforms a vector or tensor-valued quantity based on the current configuration to the reference configuration.

Polar decomposition

At each point $X \in \Omega_0$ and each time t, we have the following unique polar decomposition of the deformation gradient F

$$\mathbf{F} = \mathbf{R}\mathbf{U} = \mathbf{v}\mathbf{R} \;,\quad \mathbf{R}^T\mathbf{R} = \mathbf{I} \;,\quad \mathbf{U} = \mathbf{U}^T \;,\quad \mathbf{v} = \mathbf{v}^T$$

where **U** and **v** define unique, positive definite, symmetric tensor, which we call the **right (or material) stress tensor** and the **left (or spatial) stretch tensor**, respectively. The unique **R** is a proper orthogonal tensor, with det**R** = 1, called the **rotation tensor**.

The positive definite and symmetric tensors **U** and **v** are introduced, so that

$$\mathbf{U}^2 = \mathbf{UU} = \mathbf{C} \quad \text{and} \quad \mathbf{v}^2 = \mathbf{vv} = \mathbf{b}$$

Eigenvalues and eigenvectors of strain tensors

We introduce the mutually orthogonal and normalized set of eigenvectors $\{\hat{N}_i\}$ and their corresponding eigenvalues λ_i, i = 1, 2, 3, of the material tensor **U**

$$\mathbf{U}\,\hat{N}_i = \lambda_i\,\hat{N}_i \quad \text{with } |\hat{N}_i| = 1$$

Furthermore we obtain the eigenvalues problem for C

$$\mathbf{C}\,\hat{N}_i = \mathbf{U}^2\,\hat{N}_i = \lambda_i^2\,\hat{N}_i$$

The eigenvalues of the symmetric tensor **U** are λ_i are called the principal stretches, while for the symmetric tensor **C** we find the squares of the principal stretch denoted λ_i^2.

4. Stress tensors

Real stress tensor in the deformed configuration (Newton's third law) is described by Cauchy stress tensor $\boldsymbol{\sigma}$. Stress tensor measure in undeformed configuration can be described according to first Piola-Kirchhoff stress tensor **P** (nominal stress tensor).

The relation between the **Cauchy stress tensor** and **first Piola-Kirchhoff stress tensor** is

$$\boldsymbol{\sigma}(x, t)nds = \mathbf{P}(X, t)\, N\, dS \tag{12}$$

By means of Piola transformation the relations between **P** and $\boldsymbol{\sigma}$ may be written in the form

$$\mathbf{P} = J\boldsymbol{\sigma}\mathbf{F}^{-T}, \quad \boldsymbol{\sigma} = J^{-1}\,\mathbf{P}\,\mathbf{F}^{T} \tag{13}$$

The stress tensor work conjugate with the Lagrange strain is the **second Piola-Kirchhoff stress tensor S** which is related to the Cauchy stresses

$$\mathbf{S} = J\mathbf{F}^{-1}\boldsymbol{\sigma}\,\mathbf{F}^{-T} = \mathbf{F}^{-1}\,\mathbf{P} = \mathbf{S}^{T} \tag{14}$$

with its inverse

$$\boldsymbol{\sigma} = J^{-1}\mathbf{F}\,\mathbf{S}\,\mathbf{F}^{T} \tag{15}$$

From eq.(14) we find a fundamental relationship between the first Piola-Kichhoff stress tensor **P** and symmetric second Piola-Kichhoff stress tensor **S**, i.e.

$$\mathbf{P} = \mathbf{FS} \tag{16}$$

5. Constitutive equations

5.1 Hyperelastic material

The constitutive equations in large strain elasticity or hyperelasticity are defined in terms of a strain-energy function. A so-called **hyperelastic material** postulates the existence of a **Helmholtz free-energy function Ψ**, which is defined per unit reference volume rather than unit mass.

We now restrict attention to **homogeneous** materials. For this type of ideal material the strain energy function depends upon only the deformation gradient **F**, $\Psi = \Psi(\mathbf{F})$, the Helmholtz free-energy function is referred to as the **strain-energy function**. For continuum bodies which may undergo **inhomogeneous** deformation, in which $\Psi = \Psi(\mathbf{F}, \mathbf{X})$.

The stress responses of hyperelastic materials is derived from a given scalar-valued strain energy function

$$\mathbf{P} = \frac{\partial \Psi(\mathrm{F})}{\partial \mathrm{F}} \quad , \tag{17}$$

$$\boldsymbol{\sigma} = \mathrm{J}^{-1} \frac{\partial \Psi(\mathrm{F})}{\partial \mathrm{F}} \mathbf{F}^{\mathrm{T}} = \mathrm{J}^{-1} \mathbf{F} \left(\frac{\partial \Psi(\mathrm{F})}{\partial \mathrm{F}} \right)^{\mathrm{T}} , \tag{18}$$

where **P** is first Piola-Kirchhoff stress tensor, $\boldsymbol{\sigma}$ is Cauchy stress tensor, and J is determinant of deformation gradient (or volume ratio).

We have **equivalent forms** of the strain energy function [8]

$$\Psi(\mathbf{F}) = \Psi(\mathbf{C}) = \Psi(\mathbf{E}) \tag{19}$$

By means of the chain rule and some mathematical transformations can be given an important reduced form of the **constitutive equation for hyperelastic materials**, namely

$$\boldsymbol{\sigma} = \mathrm{J}^{-1} \mathbf{F} \left(\frac{\partial \Psi(\mathrm{F})}{\partial \mathrm{F}} \right)^{\mathrm{T}} = 2\mathrm{J}^{-1} \mathbf{F} \frac{\partial \Psi(\mathrm{C})}{\partial \mathrm{C}} \mathbf{F}^{\mathrm{T}} \quad , \tag{20}$$

Alternative expressions may be obtained for the first Piola-Kirchhoff stress tensor and second Piola-Kirchhoff stress tensor as

$$\mathbf{P} = 2\mathbf{F} \frac{\partial \Psi(\mathrm{C})}{\partial \mathrm{C}} , \tag{21}$$

$$\mathbf{S} = 2 \frac{\partial \Psi(\mathrm{C})}{\partial \mathrm{C}} = \frac{\partial \Psi(\mathrm{E})}{\partial \mathrm{E}} , \tag{22}$$

5.2 Isotropic hyperelastic materials

Constitutive equations of **isotropic hyperelastic materials** may be expressed in terms of the principal invariants if strain energy function must be an invariant under a rotation. Having this in mind, the strain-energy functions may be expressed as a set of independent strain invariants of the symmetric Cauchy-Green tensors **C** and **b** that

$$\Psi = \Psi[\mathrm{I}_1(\mathbf{C}), \mathrm{I}_2(\mathbf{C}), \mathrm{I}_3(\mathbf{C})] = \Psi[\mathrm{I}_1(\mathbf{b}), \mathrm{I}_2(\mathbf{b}), \mathrm{I}_3(\mathbf{b})] \tag{23}$$

If the strain-energy function Ψ is an invariant, we may regard Ψ as a function of the principal stretches λ_i, i = 1, 2, 3

$$\Psi(\mathbf{C}) = \Psi(\lambda_1, \lambda_2, \lambda_3) \tag{24}$$

The principal Cauchy stresses σ_i and the three Piola-Kirchhoff stresses P_i and S_i, i = 1, 2, 3 are obtained as

$$\sigma_i = \mathrm{J}^{-1} \lambda_i \frac{\partial \Psi}{\partial \lambda_i} \quad , \mathrm{P}_i = \frac{\partial \Psi}{\partial \lambda_i} \quad , \quad \mathrm{S}_i = \frac{1}{\lambda_i} \frac{\partial \Psi}{\partial \lambda_i} \tag{25}$$

5.3 Incompressible Hyperelastic Materials

Numerous polymeric materials can sustain finite strains without noticeable volume changes. Such types of material may be regarded as **incompressible** so that only isochoric motions are possible. For many cases, this is a common idealization and accepted assumption often invoked in continuum and computational mechanics.

Incompressible hyperelastic materials are characterized by the incompressibility constraint

$$\mathrm{J} = 1 \tag{26}$$

In order to derive general constitutive equations for incompressible hyperelastic materials, we may postulate the strain-energy function

$$\Psi = \Psi(\mathbf{F}) - p(J - 1) \tag{27}$$

where the scalar p is an indeterminate Lagrange multiplier, which can be identified as a **hydrostatic pressure**. Note that the scalar may only be determined from the equilibrium equations and the boundary conditions.

General constitutive equations for first and second Piola-Kichhoff stress tensors and Cauchy stress tensor may be expressed as

$$\mathbf{P} = - p\mathbf{F}^{-T} + \frac{\partial \Psi(F)}{\partial F} \tag{28}$$

$$\mathbf{S} = - p\mathbf{F}^{-1}\,\mathbf{F}^{-T} + \mathbf{F}^{-1}\frac{\partial \Psi(F)}{\partial F} = - p\mathbf{C}^{-1} + 2\frac{\partial \Psi(C)}{\partial C} \tag{28}$$

$$\boldsymbol{\sigma} = - p\,\mathbf{I} + \frac{\partial \Psi(F)}{\partial F}\,\mathbf{F}^{T} = - p\mathbf{I} + \mathbf{F}\left(\frac{\partial \Psi(F)}{\partial F}\right)^{T} \tag{30}$$

For a view on the theory of incompressible hyperelasticity see. For example OGDEN [1982, 1986]. A suitable strain energy function for incompressible isotropic hyperelastic material is given by

$$\Psi = \Psi[I_1(\mathbf{C}), I_2(\mathbf{C})] - \frac{1}{2}p(I_3 - 1) \tag{31}$$

where p/2 is an indeterminate Lagrange multiplier.

5.4 Compressible Hyperelastic Materials

A material which can undergo changes of volume is said to be **compressible**. The only restriction on this class of materials that the volume ratio J must be positive (J>0). Since some materials behave quite differently in bulk and shear it is most beneficial to split the deformation locally into a so-called **volumetric part** and an **isochoric part**. The deformation gradient **F** and right Cauchy-Green strain tensor **C** may be divided into **volume-changing (dilatational)** and **volume-preserving (distortational)** parts, often use in elastoplasticity

$$\mathbf{F} = (J^{1/3}\,\mathbf{I})\;\overline{\mathbf{F}} \quad , \quad \mathbf{C} = \mathbf{F}^{T}\,\mathbf{F} = (J^{2/3}\,\mathbf{I})\,\overline{\mathbf{C}} \;, \tag{32}$$

The terms $J^{1/3}\,\mathbf{I}$ and $J^{2/3}\,\mathbf{I}$ are associated with volume-changing deformations, while $\overline{\mathbf{F}}$ and $\overline{\mathbf{C}} = \overline{\mathbf{F}}^{T}\,\overline{\mathbf{F}}$ are associated with volume-preserving deformation of the material (called **modified deformation gradient** and **modified right Cauchy-Green tensor**), with $\det\overline{\mathbf{F}} = \overline{\lambda}_1\,\overline{\lambda}_2\,\overline{\lambda}_3 = 1$ ($\overline{\lambda}_i = J^{-1/3}\lambda_i$) and $\det\overline{\mathbf{C}} = 1$.

Similarly the strain-energy function may be written as

$$\Psi = \Psi_{vol}(J) + \Psi_{iso}(\overline{\mathbf{C}}) \tag{33}$$

where $\Psi_{vol}(J)$ and $\Psi_{iso}(\overline{\mathbf{C}})$ are given scalar-valued functions of J and $\overline{\mathbf{C}}$, which describe the so-called volumetric elastic response and isochoric elastic response of the material, respectively.

5.5 Transversely isotropic Materials

Numerous materials are composed of a matrix material and one or more families of fibers, are called **composite materials** (or fiber-reinforced composites). The types of composites have strong directional properties and their mechanical responses are regarded as **anisotropic**. The simplest anisotropic material is reinforced by only one family of fibers and has single preferred direction. we call **transversely isotropic material.** The material response along orthogonal to this preferred direction is isotropic.

Because of the directional dependence on the deformation, expressed by the unit vector field $\mathbf{a}_0$, we require that free energy depends explicitly on both the right Cauchy-Green tensor $\mathbf{C}$ and the fiber direction a in the reference configuration

$$\Psi = \Psi(\mathbf{C}, \mathbf{a}_0 \otimes \mathbf{a}_0) = \Psi\,(\mathbf{C}, \mathrm{A}) \tag{34}$$

As above mentioned an isotropic hyperelastic material may be represented by the three invariants I_1, I_2, I_3 of either $\mathbf{C}$ or $\mathbf{b}$. For a transversely hyperelastic material, the free energy can be written in terms of five independent scalar invariants as

$$\Psi = \Psi[I_1(\mathbf{C}), I_2(\mathbf{C}), I_3(\mathbf{C}), I_4(\mathbf{C}, \mathbf{a}_0), I_5(\mathbf{C}, \mathbf{a}_0)] \tag{35}$$

where I_4, I_5 are so-called **pseudo-invariants** of $\mathbf{C}$ and $\mathbf{a}_0 \otimes \mathbf{a}_0$, which given by

$$I_4(\mathbf{C}, \mathbf{a}_0) = \mathbf{a}_0 \cdot \mathbf{C}\,\mathbf{a}_0 = \lambda^2 \;\; ; \;\; I_5(\mathbf{C}, \mathbf{a}_0) = \mathbf{a}_0 \cdot \mathbf{C}^2\,\mathbf{a}_0 \tag{36}$$

The two pseudo-invariants I_4, I_5 arise directly from the anisotropy and contribute to the free energy. They describe the properties of the fiber family and its interaction with the other material constituents.

5.6 Incompressible transversely isotropic materials

We now consider transversely isotropic materials with an incompressible isotropic matrix material, i.e. $I_3 = 1$.

Firstly, in case the embedded fibers are extensible we can be able postulate a free-energy in terms of remaining four independent invariants. In view of (31) we have assumption

$$\Psi = \Psi[I_1(\mathbf{C}), I_2(\mathbf{C}), I_4(\mathbf{C}, \mathbf{a}_0), I_5(\mathbf{C}, \mathbf{a}_0)] - \frac{1}{2}\,p(I_3 - 1) \tag{37}$$

where p/2 is an indeterminate Lagrange multiplier.

Secondly, we study an incompressible isotropic matrix material which is continuously reinforced throughout by inextensible fibers. This mean that $\lambda = 1$ and the fourth invariant equals to one (I_4=1). By adding the term $q(I_4 - 1)/2$ to the free-energy we obtain the function

$$\Psi = \Psi[I_1(\mathbf{C}), I_2(\mathbf{C}), I_5(\mathbf{C}, \mathbf{a}_0)] - \frac{1}{2}\,p(I_3 - 1) - \frac{1}{2}\,q(I_4 - 1) \tag{38}$$

where q/2 is an additional indeterminate Lagrange multiplier.

The constitutive equations with an incompressible isotropic matrix material and inextensible fibers are obtained as

$$\mathbf{S} = -p\mathbf{C}^{-1} - q\mathbf{a}_0 \otimes \mathbf{a}_0 + 2\left(\frac{\partial \Psi}{\partial I_1} + I_1 \frac{\partial \Psi}{\partial I_2}\right)\mathrm{I} - 2\frac{\partial \Psi}{\partial I_2}\mathbf{C} + 2\frac{\partial \Psi}{\partial I_5}(\mathbf{a}_0 \otimes \mathbf{C}\mathbf{a}_0 + \mathbf{C}\mathbf{a}_0 \otimes \mathbf{a}_0) \tag{39}$$

$$\boldsymbol{\sigma} = -p\mathbf{I} - q\mathbf{a}_0 \otimes \mathbf{a}_0 + 2\frac{\partial \Psi}{\partial I_1}\mathbf{b} - 2\frac{\partial \Psi}{\partial I_2}\mathbf{b}^{-1} + 2\frac{\partial \Psi}{\partial I_5}\,(\mathbf{a} \otimes \mathbf{b}\mathbf{a} + \mathbf{b}\mathbf{a} \otimes \mathbf{a}) \tag{40}$$

5.7 Composite materials with two families of fibers

The matrix material is assumed to be hyperelastic. The preferential fiber directions in the reference and the current configuration are denoted by the unit vector fields $\mathbf{a}_0$, $\mathbf{g}_0$ and $\mathbf{a}$, $\mathbf{g}$, respectively. For notational simplicity we have introduced the abbreviation

$$\mathbf{A}_0 = \mathbf{a}_0 \otimes \mathbf{a}_0\,, \;\; \mathbf{G}_0 = \mathbf{g}_0 \otimes \mathbf{g}_0$$

By analogy we may postulate the free energy

$$\Psi = \Psi(\mathbf{C}, \mathbf{A}_0, \mathbf{G}_0) \tag{41}$$

With five invariants in the isotropic case. For a composite materials with two families of fibers we need add four invariants. The pseudo-invariants I_4, …, I_8 are associated with the anisotropy generated by the two families of fibers.

$$I_6(\mathbf{C}, \mathbf{g}_0) = \mathbf{g}_0 \cdot \mathbf{C}\mathbf{g}_0 \,, \quad I_7(\mathbf{C}, \mathbf{g}_0) = \mathbf{g}_0 \cdot \mathbf{C}^2\mathbf{g}_0 \,, \quad I_8(\mathbf{C}, \mathbf{a}_0, \mathbf{g}_0) = (\mathbf{a}_0 \cdot \mathbf{g}_0)\mathbf{a}_0 \cdot \mathbf{C}\mathbf{g}_0 \tag{42}$$

If $\mathbf{a}_0 \cdot \mathbf{g}_0 = 0$, the two families of fibers have orthogonal directions. Then, the material is said to be **orthotropic** in the reference configuration and the free energy has the form $\Psi = \Psi(I_1,\ldots, I_7)$.

A further case may be found under the assumption that the isotropic matrix material is incompressible, i.e. $I_3 = 1$. Additionally, the families of fibers may be also inextensible in the two fiber directions $\mathbf{a}_0$ and $\mathbf{g}_0$, consequently $I_4 = 1$ and $I_6 = 1$. For this case a suitable Helmholtz free-energy function is given by

$$\Psi = \Psi[I_1(\mathbf{C}), I_2(\mathbf{C}), I_5(\mathbf{C}, \mathbf{a}_0), I_7(\mathbf{C}, \mathbf{g}_0)] - \frac{1}{2}p(I_3 - 1) - \frac{1}{2}q(I_4 - 1) - \frac{1}{2}r(I_6 - 1) \tag{43}$$

with the indeterminant Lagrange multipliers p/2, q/2, r/2

5.8 Viscoelastic materials at large strains [7]

Here we use the theory of compressible hyperelasticity within the isothermal regime. We postulate a decoupled representation of the **Helmholtz free-energy function** Ψ. The free energy uses the multiplicative decomposition of deformation gradient (**F**) into dilational (volumetric) and volume-preserving (isochoric) parts [3][7]

$$\mathbf{F} = J^{1/3}\,\overline{\mathbf{F}} \,, \quad \overline{\mathbf{F}} = \overline{\mathbf{F}}_{\text{elas}}\,\mathbf{F}_{\text{vis}}$$

The change of Ψ within an isothermal elastic process from the reference to the current configuration is given as

$$\Psi(\mathbf{C}, \Gamma_1,\ldots, \Gamma_m) = \Psi^{\infty}_{\text{vol}}(J) + \Psi^{\infty}_{\text{iso}}(\overline{\mathbf{C}}) + \sum_{\alpha=1}^{m}\Psi_{\alpha}(\overline{\mathbf{C}}, \Gamma_{\alpha}), \tag{44}$$

valid for some closed time interval $t \in [0, T]$ of interest. Above equation is applied to the **generalized Maxwell model** with arbitrary number m of Maxwell elements. In which $\Psi^{\infty}_{\text{vol}}(J)$ and $\Psi^{\infty}_{\text{iso}}(\overline{\mathbf{C}})$ are strain-energy functions per unit reference volume and characterize the equilibrium state of the solid. They can be identified as the terms volumetric hyperelastic response and the isochoric hyperelastic response of sufficiently slow processes, respectively. The dissipative potential $\sum_{\alpha=1}^{m}\Psi_{\alpha}$ is responsible for the viscoelastic contribution, called **configurational free energy** of the viscoelastic solid and characterize the non-equilibrium state, i.e. the behavior of relaxation and creep. The **strain-like internal variable $\boldsymbol{\Gamma_{\alpha}}$** characterizes the **relaxation and/or creep** behavior of the material.

We assume that each contribution to the free energy Ψ must satisfy the normalization condition

$$\Psi^{\infty}_{\text{vol}}(\mathbf{I}) = 0 \,, \quad \Psi^{\infty}_{\text{iso}}(\mathbf{I}) = 0 \quad , \quad \sum_{\alpha=1}^{m}\Psi_{\alpha}(I, I) = 0, \tag{45}$$

5.9 Visco-hyperelastic materials at large strains

Based on established approaches to the formulation of constitutive relationships, a new objective constitutive equation is proposed to describe the three-dimensional visco-hyperelastic large deformation behaviour of incompressible rubber-like materials under high strain rates. Static response is accommodated by a component comprising a hyperelastic relationship based on an

elastic strain energy potential. It is found that a three-term truncated series for this potential adequately describes hyperelasticity. Another equation component, a generalised non-linear Maxwell model, is introduced to characterise viscoelastic response under high strain rates. The total expression represents a hyperelastic solid in parallel with a generalised Maxwell model, thus characterising hyperelasticity as well as strain rate and strain history dependent viscoelasticity.

$$\Psi(\mathbf{C}, \Gamma_1,\ldots, \Gamma_m) = \Psi_{vol}^{\infty}(J) + \Psi_{iso}^{\infty}(\overline{C}) + \sum_{\alpha=1}^{m} Y_{\alpha}(\overline{C}, \Gamma_{\alpha}), \tag{46}$$

6. Some forms of strain-energy functions

Numerous specific forms of strain energy functions to describe the elastic properties of incompressible as well compressible materials have been proposed in the literature. In particular, some forms of strain energy functions which are most used for computing finite elements.

6.1 The strain energy functions for isotropic materials

6.1.1 Ogden model for incompressible materials (rubber-like, biological soft tissues)

Rubber is often regarded as **incompressible** with the constraint condition $J = \lambda_1\lambda_2\lambda_3 = 1$. **Ogden** proposed strain energy is a function of the principal stretches λ_i (i = 1, 2, 3) in the form

$$\Psi = \Psi(\lambda_1, \lambda_2, \lambda_3) = \sum_{p=1}^{N} \frac{\mu_p}{\alpha_p}\left(\lambda_1^{\alpha_p} + \lambda_2^{\alpha_p} + \lambda_3^{\alpha_p} - 3\right) \tag{47}$$

On comparison with linear theory we obtain the (consistency) condition

$$2\mu = \sum_{p=1}^{N} \mu_p\alpha_p \quad \text{with} \quad \mu_p\alpha_p > 0 \tag{48}$$

where the parameter μ denotes the classical shear modulus in the reference configuration, N is a positive which determine the number of terms in the strain energy function, μ_p are (constant) shear modulus and α_p are dimensionless constants (determine by experiment), p = 1, …, N

6.1.2 Mooney-Rivlin, Neo-Hookean, Varga model for incompressible materials (rubber-like)

As a special case we obtain from Ogden model for incompressible materials.

For example, the very useful **Mooney-Rivlin model** obtain by setting N = 2, $\alpha_1 = 2$, $\alpha_2 = -2$

$$\begin{aligned}\Psi &= c_1(\lambda_1^2 + \lambda_2^2 + \lambda_3^2 - 3) + c_2(\lambda_1^{-2} + \lambda_2^{-2} + \lambda_3^{-2} - 3)\\ &= c_1(I_1 - 3) + c_2(I_2 - 3)\end{aligned} \tag{49}$$

with the constants $c_1 = \mu_1/2$ and $c_2 = \mu_2/2$ ($\mu = \mu_1 - \mu_2$).

This model is often employed in the description of the behavior of isotropic rubber-like materials.

The **Neo-Hookean model** obtain by replace N = 1, $\alpha_1 = 2$

$$\Psi = c_1(\lambda_1^2 + \lambda_2^2 + \lambda_3^2 - 3) = c_1(I_1 - 3) \tag{50}$$

with the constant $c_1 = \mu_1/2$ and the shear modulus $\mu = \mu_1$ according to (48).

At last special case of Ogden's model we introduce the model by **Varga**, which obtain by setting N = 1, $\alpha_1 = 1$, i.e.

$$\Psi = c_1(\lambda_1 + \lambda_2 + \lambda_3 - 3) \tag{51}$$

with the constant $c_1 = \mu_1/2$ and the shear modulus $\mu = \mu_1/2$ according to (48)

6.1.3 Yeoh, Arruda and Boyce model for incompressible (rubber-like) materials [7]

Nearly all practical engineering elastomers contain reinforcing fillers such as **carbon black filled rubbers**, which have typical dimensions of the order of $1.0 – 2.0.10^{-2}$m.

Consider a simple shear deformation of a filler-loaded rubber. Physical observation show that the shear modulus μ of the material caries with deformation in a significant way. To be more specific, μ decreases with increasing deformation initially and then rises again at large deformations.

Experiments shown that $\frac{\partial \Psi}{\partial I_2}$ is numerically close to zero. **Yeoh** made a simplifying assumption that $\frac{\partial \Psi}{\partial I_2} = 0$ and proposed a three-term strain-energy function where the second strain invariant does nor appear

$$\Psi = c_1(I_1 - 3) + c_2(I_1 - 3)^2 + c_3(I_1 - 3)^3 \tag{52}$$

The shear modulus μ involves first-order and second-order terms in $(I_1 - 3)$ and approximates the observed nonlinear physical behavior with satisfying accuracy (provided $c_2<0$ and $c_1>0$, $c_3>0$)

$$\mu = 2c_1 + 4c_2(I_1 - 3) + 6c_3(I_1 - 3)^2 > 0 \tag{53}$$

Another material model for the response of rubber which has similar structure to Yeoh model is due to **Arruda and Boyce**. The strain-energy function is derived from the inverse Langevin function by means of Taylor's expansion

$$\Psi = \mu\left[\frac{1}{2}(I_1 - 3) + \frac{1}{20n}(I_1^2 - 9) + \frac{11}{1050n^2}(I_1^3 - 27) + ...\right] \tag{54}$$

where μ denotes the shear modulus and n is the number of segments (each of the same length) in a chain.

6.1.4 Blatz and Ko model for foamed elastomers [7]

The strain-energy function combined theoretical arguments and experimental data

$$\Psi(I_1, I_2, I_3) = f\frac{\mu}{2}\left[(I_1 - 3) + \frac{1}{\beta}(I_3^{-\beta} - 1)\right] + (1 - f)\frac{\mu}{2}\left[\left(\frac{I_2}{I_3} - 3\right) + \frac{1}{\beta}(I_3^{\beta} - 1)\right] \tag{55}$$

in which μ and ν ($\beta = \frac{\nu}{1-2\nu}$) denote the shear modulus and Poisson's ratio, and $f \in [0,1]$ is an interpolation parameter.

6.1.5 Saint-Venant Kirchhoff model [2][7]

This model is characterized by the strain-energy function

$$\Psi(\mathbf{E}) = \frac{\gamma}{2}(\mathrm{tr}\mathbf{E})^2 + \mu\mathrm{tr}\mathbf{E}^2 \tag{56}$$

in which $\gamma > 0$ and $\mu > 0$ are two constants of Lamé.

Saint-Venant Kirchhoff model is a classical nonlinear model for compressible hyperelastic materials. Note that this material model is suitable for large displacement but it is not recommended to use for large compressive strains.

6.2 The strain energy functions for anisotropic materials

6.2.1 Transversely isotropic materials

The strain-energy functions can be decomposed into fully isotropic component and transversely isotropic component as

$$\Psi = \Psi_{iso} + \Psi_{trn} \tag{57}$$

Bonet [2] used the isotropic component by the strain energy function of Saint-Venant model (56)

In order to generalize the above equation to the fully **NONLINEAR** regime, the Neo-Hookean energy Ψ_{nh} (50) can be used instead of the St.Venant function to described the isotropic component of the strain function. This gives

$$\Psi = \Psi_{nh} + \Psi_{trn} \tag{58}$$

Anisotropic part of strain-energy function is expressed in the form

$$\Psi_{trn} = [\alpha + \beta(I_1 - 3) + \gamma(I_4 - 1)] (I_4 - 1) - \frac{1}{2}\alpha(I_5 - 1) \tag{59}$$

The transversely isotropic component of the constitutive model presented above is still that of the St.Venant model and it is therefore linear with respect to the Lagrangian strain tensor. This could still cause difficulties in **large strain** applications despite the fact that the isotropic component is valid for the **fully nonlinear** range. In addition, although the material is transversely isotropic in the undeformed configuration, the resulting elasticity tensor in the deformed setting is not transversely isotropic. These two shortcomings can in fact be very easily remedied by altering the transversely isotropic component of the strain energy function by replacing in Eq.(59) the term $(I_1 - 3)$, which is linear in C, with term lnJ, thus giving

$$\Psi_{trn} = [\alpha + \beta \ln J + \gamma(I_4 - 1)] (I_4 - 1) - \frac{1}{2}\alpha(I_5 - 1) \tag{60}$$

6.2.2 Composite materials with two families of fibers

For the composite materials with two families of fibers we can be written strain-energy function as

$$\Psi = \Psi_{iso} + \Psi_{aniso} \tag{61}$$

Holzapfel and his colleagues [5-7] used the isotropic component by the strain energy function of Mooney-Rivlin (49) or Neo-Hookean (50) or other appropriate models. Anisotropic part of strain-energy function is expressed in the form

$$\Psi_{aniso} = \frac{k_1}{2k_2} \sum_{i=4,6} \left\{ \exp\left[k_2 (I_i - 1)^2 \right] - 1 \right\} \tag{62}$$

where $k_1>0$ is stress-like material parameter and $k_2>0$ is a dimensionless parameter.

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